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EXTENDED METHOD OF QUASILINEARIZATION FOR A THREE-POINT BOUNDARY VALUE PROBLEM WITH GENERAL NONLINEAR NONCONVEX BOUNDARY CONDITIONS

*Ahmed Alsaedi** and *Mohammed Alhuthali*†
Department of Mathematics,
Faculty of science, King Abdul Aziz University,
P.O.Box. 80257, Jeddah 21589, Saudi Arabia

Abstract

In this paper, We apply the generalized quasilinearization technique to obtain a monotone sequence of approximate solutions converging monotonically and quadratically to the unique solution of a second order three-point boundary value problem with general nonlinear nonconvex boundary conditions.

Key Words and Phrases: Quasilinearization, three-point boundary value problem, quadratic convergence.
AMS Subject Classifications (2000): 34B10, 34B15.

1 Introduction

The method of quasilinearization initiated by Bellman and Kalaba [1], and generalized by Lakshmikantham [2-3] has been studied and extended in several diverse disciplines. In fact, it is generating a rich history and a comprehensive description of this method can be found in [4-10].
Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [11], have been addressed by many authors, for example, [12-14]. In particular, Eloe and Gao [15] discussed the quasilinearization method for a three-point boundary value problem.
The aim of this paper is to relax the assumption of convexity/concavity on the nonlinear general boundary conditions involved in the second order three-point boundary value problem and discuss the extended method of quasilinearization for this problem. In fact, we develop a sequence of approximate solutions converging monotonically and quadratically to a solution of the problem.

*E-mail address: aalsaedi@hotmail.com
†E-mail address: moadth@yahoo.com

2 Preliminary Results

We consider the three-point boundary value problem with general nonlinear boundary conditions

$$x''(t) = f(t,x(t)), \tag{1}$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = g(x(\frac{1}{2})), \tag{2}$$

where $f \in C[[0,1] \times R, R]$, $p,q > 0$ with $p > 1$ and $g : R \longrightarrow R$ is continuous. By Green's function method, the solution, $x(t)$ of (1)-(2) can be written as

$$x(t) = a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + g(x(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + \int_0^1 G(t,s)f(s,x(s))ds,$$

where the Green's function $G(t,s)$ for the general three-point boundary value problem is given by

$$G(t,s) = \frac{1}{(p^2+2pq)} \begin{cases} (pt+q)(p(s-1)-q), & \text{if } 0 \le t \le s \le 1 \\ (p(t-1)-q)(ps+q), & \text{if } 0 \le s \le t \le 1 \end{cases}$$

Notice that $G(t,s) < 0$ on $(0,1) \times (0,1)$.

We say that $\alpha \in C^2[0,1]$ is a lower solution of the boundary value problem (1)-(2) if

$$\alpha''(t) \ge f(t,\alpha),\ t \in [0,1],$$

$$p\alpha(0) - q\alpha'(0) \le a, \qquad p\alpha(1) + q\alpha'(1) \le g(\alpha(\frac{1}{2})),$$

and $\beta \in C^2[0,1]$ be an upper solution of the boundary value problem (1)-(2) if

$$\beta''(t) \le f(t,\beta),\ t \in [0,1],$$

$$p\beta(0) - q\beta'(0) \ge a, \qquad p\beta(1) + q\beta'(1) \ge g(\beta(\frac{1}{2})).$$

Now, we state the following theorems which play a pivotal role in the proof of the main result (for the proof of these theorems, see [16]).

Theorem 1. Assume that f is continuous with $f_x > 0$ on $[0,1] \times R$ and g is continuous with $0 \le g' < 1$ on R. Let β and α be the upper and lower solutions of (1)-(2) respectively. Then $\alpha(t) \le \beta(t)$, $t \in [0,1]$.

Theorem 2. Assume that f is continuous on $[0,1] \times R$ and g is continuous on R satisfying $0 \le g' < 1$. Further, we assume that there exist an upper solution β and a lower solution α of (1)-(2) such that $\alpha(t) \le \beta(t)$, $t \in [0,1]$. Then there exists a solution $x(t)$ of (1)-(2) satisfying $\alpha(t) \le x(t) \le \beta(t)$, $t \in [0,1]$.

3 Main Result

Theorem 3. Assume that

($\mathbf{A_1}$) $f(t,x) \in C([0,1] \times R)$ such that $\frac{\partial f}{\partial x}(t,x) > 0$, $\frac{\partial^2}{\partial x^2}(f(t,x)+\phi(t,x)) \leq 0$, where $\frac{\partial^2}{\partial x^2}\phi(t,x) \leq 0$ for some continuous function $\phi(t,x)$.

($\mathbf{A_2}$) $\alpha, \beta \in C^2[0,1]$ are lower and upper solutions of (1.1)-(1.2) respectively.

($\mathbf{A_3}$) $g(x), g'(x)$ are continuous on R with $0 \leq g' \leq 1$ and $g''(x) + \psi''(x) \geq 0$ for some continuous function $\psi(x)$ satisfying $\psi''(x) \geq 0$.

Then there exists a monotone sequence of solutions $\{w_n\}$ converging quadratically to the unique solution, x of (1)-(2).

Proof. Define $F : [0,1] \times R \longrightarrow R$ by

$$F(t,x) = f(t,x) + \phi(t,x).$$

Using (A_1) and (A_3), we obtain

$$f(t,x) \leq F(t,y) + F_x(t,y)(x-y) - \phi(t,x), \tag{3}$$

and

$$g(x) \geq \chi(y) + \chi'(y)(x-y) - \psi(x), \tag{4}$$

where $\chi(x) = g(x) + \psi(x)$ and $\alpha \leq y \leq x \leq \beta$.
Define

$$F^*(t,x,y) = F(t,y) + F_x(t,y)(x-y) - \phi(t,x),$$

and

$$h(x,y) = \chi(y) + \chi'(y)(x-y) - \psi(x).$$

We observe that

$$f(t,x) = \min_y F^*(t,x,y), \qquad f(t,x) = F^*(t,x,x) \tag{5}$$

$$g(x) = \max_y h(x,y), \qquad g(x) = h(x,x). \tag{6}$$

In view of the fact that $f_x(t,x) > 0$, we find that $F_x^*(t,x,y) > 0$ which implies that $F_x^*(t,x,y)$ is increasing in x for each $(t,y) \in [0,1] \times R$. Similarly, $g'(x) = h_x(x,y)$ and by (A_3), we have $0 \leq h_x(x,y) \leq 1$. Select $\alpha = w_0$ and consider the following BVP

$$x''(t) = F^*(t, x(t), w_0(t)), \; t \in [0,1], \tag{7}$$

$$px(0) - qx'(0) = a, \qquad px(1) + qx'(1) = h(x(\tfrac{1}{2}), w_0(\tfrac{1}{2})). \tag{8}$$

Using (A_3), (5) and (6), we obtain

$$w_0'' \geq f(t, w_0) = F^*(t, w_0, w_0), \; t \in [0,1],$$

$$pw_0(0) - qw_0'(0) \leq a, \qquad pw_0(1) + qw_0'(1) \leq g(w_0(\tfrac{1}{2})) = h(w_0(\tfrac{1}{2}), w_0(\tfrac{1}{2})),$$

and

$$\beta'' \leq f(t,\beta) \leq F^*(t, \beta, w_0), \; t \in [0,1],$$

$$p\beta(0)-q\beta'(0)\geq a, \qquad p\beta(1)+q\beta'(1)\geq g(\beta(\frac{1}{2}))\geq h(\beta(\frac{1}{2}),w_0(\frac{1}{2})),$$

which imply that w_0 and β are lower and upper solutions of (7)-(8) respectively. It follows by Theorems 1 and 2 that there exists a unique solution, w_1 of (7)-(8) such that

$$w_0(t)\leq w_1(t)\leq \beta(t),\ t\in[0,1].$$

Now, we consider the BVP

$$x''=F^*(t,x(t),w_1(t)),\ t\in[0,1], \tag{9}$$

$$px(0)-qx'(0)=a, \qquad px(1)+qx'(1)=h(x(\frac{1}{2}),w_1(\frac{1}{2})). \tag{10}$$

Again, using (A_3), (5) and (6), we get

$$w_1''=F^*(t,w_1,w_0)\geq F^*(t,w_1,w_1),\ t\in[0,1],$$

$$pw_1(0)-qw_1'(0)\leq a, \qquad pw_1(1)+qw_1'(1)=h(w_1(\frac{1}{2}),w_0(\frac{1}{2}))\leq h(w_1(\frac{1}{2}),w_1(\frac{1}{2})),$$

and

$$\beta''\leq f(t,\beta)\leq F^*(t,\beta,w_1),\ t\in[0,1],$$

$$p\beta(0)-q\beta'(0)\geq a, \qquad p\beta(1)+q\beta'(1)\geq g(\beta(\frac{1}{2}))\geq h(\beta(\frac{1}{2}),w_1(\frac{1}{2})),$$

implying that w_1 and β are lower and upper solutions of $(9)-(10)$ respectively. By the earlier arguments, there exists a solution, w_2 of $(9)-(10)$ such that

$$w_1(t)\leq w_2(t)\leq \beta(t),\ t\in[0,1].$$

Continuing this process successively, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t)\leq w_1(t)\leq w_2(t)\leq \ldots \leq w_n(t)\leq \beta(t),\ t\in[0,1],$$

where each element w_n of the sequence is a solution of the BVP

$$x''=F^*(t,x(t),w_{n-1}(t)),\ t\in[0,1],$$

$$px(0)-qx'(0)=a, \qquad px(1)+qx'(1)=h(x(\frac{1}{2}),w_{n-1}(\frac{1}{2})),$$

and is given by

$$\begin{aligned} w_n(t) &= a(\frac{-t}{p+2q}+\frac{p+q}{p^2+2pq})+h(w_n(\frac{1}{2}),w_{n-1}(\frac{1}{2}))[\frac{t}{p+2q}+\frac{q}{p^2+2pq}] \\ &+ \int_0^1 G(t,s)F^*(s,w_n(s),w_{n-1}(s))ds. \end{aligned} \tag{11}$$

Employing the fact that $[0,1]$ is compact and the monotone convergence is pointwise, it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence, then passing onto the limit $n \to \infty$, (11) gives

$$
\begin{aligned}
x(t) &= a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + h(x(\frac{1}{2}),x(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\
&+ \int_0^1 G(t,s)F^*(s,x(s),x(s))ds \\
&= a(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + g(x(\frac{1}{2}))[\frac{t}{p+2q} + \frac{q}{p^2+2pq}] \\
&+ \int_0^1 G(t,s)f(s,x(s))ds.
\end{aligned}
$$

Hence, $x(t)$ is the solution of (1)-(2).
Now, we show that the convergence of the sequence of iterates is of order $k (k \geq 2)$. For that, we define $e_n(t) = x(t) - w_n(t)$, $t \in [0,1]$ and note that $e_n(t) \geq 0$. Further

$$
pe_n(0) - qe_n'(0) = 0,\ pe_n(1) + qe_n'(1) = g(x(\frac{1}{2})) - h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})).
$$

Using the generalized mean value theorem, we have

$$
\begin{aligned}
e_{n+1}''(t) &= x'' - w_{n+1}'' \\
&= F(t,x) - \phi(t,x) - [F(t,w_n) + F_x(t,w_n)(w_{n+1} - w_n) - \phi(t,w_{n+1})] \\
&= F_x(t,c_1)(x - w_n) - F_x(t,w_n)(x - w_n) \\
&+ [F_x(t,w_n) - \phi_x(t,c_2)](x - w_{n+1}) \\
&= F_{xx}(t,c_3)(c_1 - w_n)(x - w_n) + [F_x(t,w_n) - \phi_x(t,c_2)](x - w_{n+1}) \\
&\geq F_{xx}(t,c_3)e_n^2 + f_x(t,c_2)e_{n+1} \\
&\geq -M\|e_n\|^2,
\end{aligned}
$$

where M is a bound on $F_{xx}(t,x)$ for $t \in [0,1]$, $w_n < c_3 < c_1 < x$, $w_{n+1} < c_2 < x$ and $\|e_n\| = \max\{|e_n(t)| : t \in [0,1]\}$. Thus, we have

$$
\begin{aligned}
e_{n+1}(t) &= [g(x(\frac{1}{2})) - h(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2}))](\frac{t}{p+2q} + \frac{q}{p^2+2pq}) + \int_0^1 G(t,s)e_{n+1}''(s)ds \\
&\leq [\chi(x(\frac{1}{2})) - \chi(w_n(\frac{1}{2})) - (\psi(x(\frac{1}{2})) - \psi(w_{n+1}(\frac{1}{2})))) \\
&- \chi'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))][\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + M\|e_n\|^2 \int_0^1 |G(t,s)|ds \\
&= [\chi'(c_4)(x(\frac{1}{2}) - w_n(\frac{1}{2})) - \psi'(c_5)(x(\frac{1}{2}) - w_{n+1}(\frac{1}{2})) \\
&- \chi'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))][\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + M\|e_n\|^2 \int_0^1 |G(t,s)|ds \\
&\leq [\chi''(c_6)e_n^2(\frac{1}{2}) + g'(w_n(\frac{1}{2}))e_{n+1}(\frac{1}{2})][\frac{t}{p+2q} + \frac{q}{p^2+2pq}] + M_1\|e_n\|^2,
\end{aligned}
$$

where M_1 provides a bound for $M\int_0^1 |G(t,s)|ds$, $w_n(\frac{1}{2}) < c_6 < c_4 < x(\frac{1}{2})$, $w_n(\frac{1}{2}) < c_5 < x(\frac{1}{2})$. Taking maximum over the interval $[0,1]$, we get

$$
\|e_{n+1}\| < M_5\|e_n\|^2 + \lambda_1\|e_{n+1}\|, \tag{12}
$$

where M_2 provides a bound for $|\chi''|$ on $[w_n(\frac{1}{2}), x(\frac{1}{2})]$, $|g'| \leq \lambda < 1$, $M_5 = M_4 + M_1$, $M_4 = M_2 M_3$, $\lambda_1 = \lambda M_3$ and $M_3 = \frac{1}{p+2q} + \frac{q}{p^2+2pq}$. Solving (12) algebraically, we obtain

$$\|e_{n+1}\| \leq \frac{M_5}{1-\lambda_1} \|e_n\|^2.$$

This completes the proof.

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