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[2] D.O. Hebb, Organization of Behaviour, Wiley, New York, 1949.

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On Common Fixed Point and Approximation Results of Gregus Type

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Abstract

Fixed point theorems of Ciric [3], Fisher and Sessa [4], Gregus [5], Jungck [10] and Mukherjee and Verma [17] are generalized to a locally convex space. As applications, common fixed point and invariant approximation results for subcompatible maps are obtained. Our results unify and generalize various known results to a more general class of noncommuting mappings.

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1. Introduction and preliminaries

In the sequel, (E,τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \cap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, froms a base of neighborhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with property that the seminorm $max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E,τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see[16, page 203]).

The following construction will be crucial. Suppose that M is τ-bounded subset of E. For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly $B = \cap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed absolutely convex and contains M. The linear span E_B of B in E is $\cup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$ (for details see [16,25]).

Let M be a subset of a locally convex space (E, τ). Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called I-Lipschitz if there exists $k \geq 0$ such that $p_\alpha(Tx - Ty) \leq k p_\alpha(Ix - Iy)$ for any $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$. If $k < 1$ (respectively, $k = 1$), then T is called an I-contraction (respectively, I-nonexpansive). A point $x \in M$ is a common fixed point of I and T if $x = Ix = Tx$. The set of fixed points of T is denoted by $F(T)$. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$, (2) R-weakly commuting if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists $R > 0$ such that $p_\alpha(ITx - TIx) \leq R p_\alpha(Ix - Tx)$. If $R = 1$, then the maps are called weakly commuting [20]; (3) compatible [10,11,22] if for all $p_\alpha \in A^*(\tau)$, $\lim_n p_\alpha(TIx_n - ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called (4) R-subcommuting on M (see [21]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq \frac{R}{k} p_\alpha(((1-k)q + kTx) - Ix)$ for each $k \in (0, 1)$. If $R = 1$, then the maps are called 1-subcommuting [7]; (5) R-subweakly commuting on M (see [8,9]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq R d_{p_\alpha}(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$. It is well known that R-weakly commuting, R-subcommuting and R-subweakly commuting maps are compatible but not conversely in general (see [10-12]).

If $u \in E, M \subseteq E$, then we define the set $P_M(u)$ of best M-approximants to u as $P_M(u) = \{y \in M : p_\alpha(y - u) = d_{p_\alpha}(u, M)$, for all $p_\alpha \in A^*(\tau)\}$, where $d_{p_\alpha}(u, M) = \inf\{p_\alpha(x - u) : x \in M\}$. A mapping $T : M \to E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges to 0, we have $Tx = 0$.

In [4], Fisher and Sessa obtained the following generalization of a theorem of Gregus [5].

Theorem 1.1. Let T and I be two weakly commuting mappings on a closed convex subset C of a Banach space X into itself satisfying the inequality,

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1-a)\max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \tag{1.1}$$

for all $x, y \in C$, where $a \in (0,1)$. If I is linear and nonexpansive on C and $T(C) \subseteq I(C)$, then T and I have a unique common fixed point in C.

In 1988, Mukherjee and Verma [17] replaced linearity of I by affineness in Theorem 1.1. Subsequently, Jungck [12] obtained the following generalization of Theorem 1.1 and the result of Mukherjee and Verma [17].

Theorem 1.2. Let T and I be compatible self maps of a closed convex subset C of a Banach space X. Suppose that I is continuous, linear and that $T(C) \subset I(C)$. If T and I satisfy inequality (1.1), then T and I have a unique common fixed point in C.

In this paper, we first prove that Theorems 1.1-1.2 can appreciably be extended to the setup of Hausdorff locally convex space. As applications, common fixed point and invariant approximation results for a new class of subcompatible maps are derived. Our results extend and unify the work of Al-Thagafi [1], Ciric [3], Fisher and Sessa [4], Gregus [5], Habiniak [6], Hussain and Khan [7], Hussain et al. [8], Jungck [10], Jungck and Sessa [13], Khan and Hussain [14], Khan at el. [15], Mukherjee and Verma [17], Sahab, Khan and Sessa [18], Singh [23,24] and many others.

2. Main Results

We begin with the definition of subcompatible mappings.

Definition 2.1. Let M be a q-starshaped subset of a normed space E. For the selfmaps I and T of M with $q \in F(I)$, we define $S_q(I,T) := \cup\{S(I,T_k) : 0 \leq k \leq 1\}$ where $T_k x = (1-k)q + kTx$ and $S(I,T_k) = \{\{x_n\} \subset M : lim_n Ix_n = lim_n T_k x_n = t \in M \Rightarrow lim_n \|IT_k x_n - T_k I x_n\| = 0\}$. Now I and T are subcompatible if $lim_n \|ITx_n - TIx_n\| = 0$ for all sequences $\{x_n\} \in S_q(I,T)$. We can extend this definition to locally convex space by replacing norm with a family of seminorms.

Clearly, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.2. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $I(x) = 2x - 1$ and $T(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $Iq = q$. Note that I and T are compatible. For any sequence $\{x_n\}$ in M with $lim_n x_n = 2$, we have, $lim_n Ix_n = lim_n T_{\frac{2}{3}} x_n = 3 \in M, \Rightarrow lim_n \|IT_{\frac{2}{3}} x_n - T_{\frac{2}{3}} I x_n\| = 0$. However, $lim_n \|ITx_n - TIx_n\| \neq 0$. Thus I and T are not subcompatible

$q \in F(I)$ and $T(M) \subseteq I(M)$. If the pair $\{I, T\}$ is subcompatible and satisfies, for all $p_\alpha \in A^*(\tau)$, $x, y \in M$, and all $k \in (0,1)$,

$$p_\alpha(Tx-Ty) \leq p_\alpha(Ix-Iy)+\frac{1-k}{k}\max\{d_{p_\alpha}(Ix,[q,Tx]), d_{p_\alpha}(Iy,[q,Ty])\}, \quad (2.2)$$

then I and T have a common fixed point in M provided one of the following conditions holds:
(i) M is τ-compact and T is continuous.
(ii) M is weakly compact in (E,τ), I is weakly continuous and $I-T$ is demiclosed at 0.
Proof. Define $T_n : M \to M$ by

$$T_n x = (1-k_n)q + k_n Tx$$

for some q and all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, for each n, $T_n(M) \subseteq I(M)$ as M is convex, I is linear, $Iq = q$ and $T(M) \subseteq I(M)$. Further, since the pair $\{I, T\}$ is subcompatible and I is linear with $Iq = q$ so, for any $\{x_m\} \subset M$ with $\lim_m Ix_m = \lim_m T_n x_m = t \in M$, we have

$$\begin{aligned}\lim_m p_\alpha(T_n I x_m - I T_n x_m) &= k_n \lim_m p_\alpha(TIx_m - ITx_m) \\ &= 0.\end{aligned}$$

Thus the pair $\{I, T_n\}$ is compatible on M for each n. Also, we obtain from (2.2),

$$\begin{aligned}p_\alpha(T_n x - T_n y) &= k_n p_\alpha(Tx - Ty) \\ &\leq k_n\{p_\alpha(Ix - Iy) + \frac{1-k_n}{k_n}\max\{p_\alpha(Ix - T_n x), p_\alpha(Iy - T_n y)\}\} \\ &= k_n p_\alpha(Ix - Iy) + (1-k_n)\max\{p_\alpha(Ix - T_n x), p_\alpha(Iy - T_n y)\},\end{aligned}$$

for each $x, y \in M$, $p_\alpha \in A^*(\tau)$ and $0 < k_n < 1$.
(i) M being τ-compact is τ-bounded and τ-complete. Thus by Theorem 2.6, for each $n \geq 1$, there exists an $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Now the τ-compactness of M ensures that $\{x_n\}$ has a convergent subsequence $\{x_j\}$ which converges to a point $x_0 \in M$. Since $x_j = T_j x_j = k_j T x_j + (1-k_j)$ and T is continuous, so we have, as $j \to \infty, Tx_0 = x_0$. The continuity of I implies that

$$Ix_0 = I(\lim_j x_j) = \lim_j I(x_j) = \lim_j x_j = x_0.$$

(*ii*) Weakly compact sets in (E,τ) are τ-bounded and τ-complete so again by Theorem 2.6, T_n and I have a common fixed point x_n in M for each n. The set M is weakly compact so there is a subsequence $\{x_j\}$ of $\{x_n\}$ converging weakly to some $y \in M$. The map I being weakly continuous gives that $Iy = y$. Now

$$x_j = I(x_j) = T_j(x_j) = k_j T x_j + (1 - k_j)q$$

implies that $Ix_j - Tx_j = (1-k_j)[q - Tx_j] \to 0$ as $j \to \infty$. The demiclosedness of $I - T$ at 0 implies that $(I - T)(y) = 0$. Hence $Iy = Ty = y$.

An application of Theorem 2.7 establishes the following result in best approximation theory.

Theorem 2.8. Let T and I be selfmaps of Hausdorff locally convex space (E,τ) and M a subset of E such that $T(\partial M) \subseteq M$, where ∂M denotes boundary of M and $u \in F(T) \cap F(I)$. Suppose that $P_M(u)$ is nonempty convex containing q, $q \in F(I)$, I is nonexpansive and linear on $P_M(u)$ and $I(P_M(u)) = P_M(u)$. If the pair $\{I,T\}$ is subcompatible on $P_M(u)$ and satisfies, for all $x \in P_M(u) \cup \{u\}$, $p_\alpha \in A^*(\tau)$ and $k \in (0,1)$,

$$p_\alpha(Tx - Ty)$$

$$\leq \begin{cases} p_\alpha(Ix - Iu) & \text{if } y = u, \\ p_\alpha(Ix - Iy) + \frac{1-k}{k}\max\{d_{p_\alpha}(Ix,[q,Tx]), d_{p_\alpha}(Iy,[q,Ty])\}, & \text{if } y \in P_M(u), \end{cases}$$

then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following conditions is satisfied:
(*i*) $P_M(u)$ is τ-compact and T is continuous on $P_M(u)$.
(*ii*) $P_M(u)$ is weakly compact in (E,τ), I is weakly continuous and $I - T$ is demiclosed at 0.
Proof. Let $y \in P_M(u)$. Then as in the proof of Theorem 2.6 of [15](see also [9,12]) $Ty \in P_M(u)$ which implies that T maps $P_M(u)$ into itself and the conclusion follows from Theorem 2.7.

Remark 2.9. (*i*) 1-subcommuting maps are subcompatible, consequently, Theorem 2.2-Theorem 3.3 due to Hussain and Khan [7] and Theorem 2.3 of Khan and Hussain [14] are improved and extended.
(*ii*) Commuting maps are subcompatible so Theorems 2.7-2.8 are proper generalization of the main results of Brosowski [2], Habiniak [6], Sahab et al. [18], Sahney et al. [19], Singh [23,24], Tarafdar [25], Theorems 6-7 due to Jungck and Sessa [13] and Theorem 2.6 due to Khan et al.[15].

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